[This question paper contains **4** pages]

A

Sr. No. of the Question Paper :

Name of the Course	**:**	CBCS B.Sc. (Honors) Mathematics
Unique Paper Code	**:**	32351501_OC
Name of the Paper	**:**	C-11 (Metric Spaces)
Semester	**:**	V
Duration	**:**	3 Hours **Maximum Marks :** 75

<u>Instruction for candidates</u>

1. *Question No. 1 is compulsory.* Attempt any 3 out of the remaining 5 questions.
2. *All questions carry equal marks* (18.75).
3. In the question paper, given notations have their usual meaning unless until stated otherwise. Also, if not mentioned specifically, $\mathbb{R}, \mathbb{R}^2, l_2$ or, any subset of $\mathbb{R}$ etc. will be assumed to be endowed with the usual/standard metric.

....

1 (a) State whether True or False. Justify your answer in brief.

(i) $d(x,y)=\sqrt{|x-y|}$ and $d(x,y)=(x-y)^2$ both define metrics on $\mathbb{R}$.

(ii) A function with discrete metric space as its domain is continuous.

(iii) $\mathbb{Q}\cup\{2^{1/2}\}$ is dense in $\mathbb{R}$.

(iv) Not every subset of a metric space is a metric subspace.

(v) A compact metric space is a complete metric space, but a complete metric space may not be compact metric space.

(vi) The set $[-1,0]\cup\left\{1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right\}$ is compact in $\mathbb{R}$. **2 × 6 = 12**

(b) (i) Consider a metric space (X, d) and a 1-1 function $f: X\to\mathbb{R}$. Then verify whether the following defines a metric on X: $d^*(x,y)=d(x,y)+|f(x)-f(y)|,\ x,y\in X$. **3**

(ii) Show that the set $\{x\in\mathbb{Q}: x\in[0,1]\}$ is closed in $(\mathbb{Q}, d_u)$. **1.75**

(iii) Let $A = \left\{(x,y) \in \mathbb{R}^2 : y = \sin\left(\frac{1}{x}\right), 0 < x \le 1\right\}$. Write $\overline{A}$ in $(\mathbb{R}^2, d_u)$. **2**

2 (a) (i) Prove that in a metric space (X, d) a sequence $< x_n >$ is convergent and converges to a point x in X if and only if the sequence $<d(x_n, x)>$ converges to 0 in $\mathbb{R}$. **2**

(ii) Show that the sequence $< x_n >$, where $x_n = <1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots>$ converges to $x = <1, 1/2, 1/3, \ldots, 1/n, \ldots>$ in l_2-space. **2**

(iii) Write the boundary of the set $A = (0,1) \cup \{2\}$ in $\mathbb{R}$. **1**

(b) (i) Consider the subspace $X = I \cup J$ of $\mathbb{R}$, where $I =]0,1[$ and $J = [4,7[$. Show that $d(\sup_X I, I) > 0$. **2.75**

(ii) Show that every non-empty set is a bounded metric space. **2**

(iii) Consider $\mathbb{R}$ endowed with discrete metric. Show that every singleton in it is both open and closed ball of radius less than 1. **2**

(c) (i) Show that $A = \{<x_n> \mid x_n = 0 \text{ or } 1\}$ is a metric subspace of l_∞-space. What is the induced metric on A? **3**

(ii) Give an example of a sequence in the l_∞-space which is co-ordinate wise convergent but not convergent in l_∞. **4**

3. (a) Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f: X_1 \to X_2$ be a continuous function. Define a metric d on X_1 as $d(a,b) = d_1(a,b) + d_2(f(a), f(b))$. Show that d is equivalent to d_1. Moreover, prove that $f:(X_1, d_1) \to (X_2, d_2)$ is uniformly continuous. **4+2**

(b) Let $X = C[0,1]$. Consider the metrics d and ρ defined on $C[0,1]$ by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\};$$

$$\rho(f,g) = \int_0^1 |f(x) - g(x)|dx$$

Is the identity map $i: (X,d) \to (X,\rho)$ a homeomorphism? Justify your answer. **5**

(c) Suppose $f(x,y)$ is a continuous real valued function of the two real variables x and y. Prove that the curve in the Euclidean plane whose equation is $f(x,y) = 0$ is a closed set. Is $A = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$ an open set? Justify your answer. **2+1**

(d) Is $(\mathbb{R}^2, d)$ isometric to $(\mathbb{C}, e)$, where d and e are standard metrics. Justify your answer. **3**

(e) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$. Find $d(\overline{A})$ when (i) d is usual metric on $\mathbb{R}$ (ii) d is discrete metric on $\mathbb{R}$. **1.75**

4. (a) Let (X,d) be any metric space. Does there exist a bounded metric on X which is equivalent to d? Justify your answer. **1+3**

(b) Suppose f and g be real-valued functions on a metric space (X,d). Let

$A = \{x \in X : f(x) < g(x)\}$ and $B = \{x \in X : f(x) \neq 0\}$

Is A open? Is B open? Justify your answer. **3+2**

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 1/(1+x^2)$. Is f continuous? Suppose $A \subseteq \mathbb{R}$ is a compact subset of $\mathbb{R}$. Is $f^{-1}(A)$ also compact? Give justification for your answer. **1+2**

(d) Let $X = [1,5] \cup [9,12]$ endowed with usual metric inherited from $\mathbb{R}$. Is $\overline{S(10,5)} = \bar{S}(10,5)$? Justify your answer. **3**

(e) Define $f: [0,1) \cup [2,3] \to [0,2]$ by

$$f(x) = \begin{cases} x & , \quad x \in [0,1) \\ 4-x & , \quad x \in [2,3] \end{cases}$$

Is f continuous? Is f bijective? Is f^{-1} continuous? Justify your answer. **1+1+1.75**

5. Give suitable justification to your answer.

(a) Give an example of each of the following: A function defined as $f: \mathbb{R} \to \mathbb{R}$ having (i) no fixed point; (ii) having only 1 fixed point. **2**

(b) Is $f:([0,1], d_u) \to ([0,1], d_u)$ defined by $f(x) = x^2$ a contraction map? **2**

(c) Give an example of an open set in $\mathbb{R}$ which is complete but not connected. **2**

(d) Give an example of a metric space where singletons are the only connected sets. **2**

(e) Show that in $\mathbb{R}$, the continuous image of an interval is an interval. **3**

(f) Can we have a countably infinite connected set in $\mathbb{R}$? **2**

(g) Can we have an uncountable set in $\mathbb{R}$ which is not connected? **2**

(h) Let $f:(X, d_X) \to (Y, d_Y)$ be a continuous, 1-1 and onto map, where (X, d_X) is a compact metric space. Let $A \subseteq Y$ be connected. Then what can be said about the connectedness of $f^{-1}(A)$? **2**

(i) Comment upon the connectedness of the set $S = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$. **1.75**

6. Give suitable justification to your answer.

(a) Let $X = \mathbb{R} \setminus \{0\}$. Show that there exists a continuous map f defined on X such that

$$f(x) = \begin{cases} 1, & x \in (-\infty, 0) \\ 0, & x \in (0, \infty). \end{cases}$$ **4**

(b) Can the closure of a connected set be disconnected? **2**

(c) Give an example of a metric space whose diameter is finite, but it is not compact. **2**

(d) Give an example of a countable infinite set which is not compact, but its closure is compact. **2**

(e) Give an example to justify that the continuous image of a bounded set need not be compact. **2**

(f) Give an example of a collection of closed sets whose union is closed but not compact. **2**

(g) Let $f:(X,d_X)\to(Y,d_Y)$ be a continuous functions and A be a closed and bounded subset of X. Then what can be said about the compactness of $f(A)$? **2**

(h) An arbitrary intersection of compact sets is compact? **2.75**